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A DISEQUILIBRIUM MODEL OF THE INTEREST RATE

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ABSTRACT

In the setting of a dynamic general equilibrium model we ask the following question: What happens if the interest rate is settled exogenously in a level that differs from the one which emerges from equilibria in the markets?

Although the subject of the imposition of the interest rate by an external authority on a level that differs from the so called natural interest rate has recently attracted a lot of attention in the literature, the assumption of full general equilibrium has tended to be maintained throughout. The main contribution of this paper is that we allow explicitly for disequilibrium in markets as is the tradition in other economic models when the price is settled on a level above or below the equilibrium price.

Our main conclusion is that an exogenously imposed interest rate drives the output of the economy to a level below the one that emerges from a general equilibrium without external intervention.

Keywords: Interest rate, General equilibrium, Disequilibrium.

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1. INTRODUCTION

The concept of the natural rate of interest has been recently introduced in the literature. In particular, within the New-Keynesian models, there emerged a literature that makes that concept (following the terminology of Wicksell (1898, 1906, 1907)) an essential part in the articulation of monetary policy (the main reference is Woodford, 2003)[1]. In these models the monetary natural rate of interest is defined as the rate of equilibrium real interest (variable over time in terms of real shocks on the economy) that would be obtained if prices were fully flexible (see Woodford, 2003, p . 248). It is then shown that, in the case of a simple model, the optimal monetary policy by the Central Bank (in order to keep the current output equal to the output consistent with flexible prices) consists in placing, period by period, the nominal interest rate equal to the natural rate of interest plus an eventual target value (near zero) for the inflation rate the central bank intends to achieve.

Alternatively, in this paper, we use a formal model of dynamic general equilibrium without money and with completely flexible adjustment of prices to answer another question which is: what is the difference to the economy (in terms of the level of total output and of distribution of income among economic agents) between a situation where the interest rate is determined by an external authority and another situation in which this variable is determined by the meeting of supply and demand in markets? We ask, in particular, what are the consequences for the economy of a fixed rate of interest set at a level different from the natural rate of interest (which, in the model we build, is the equilibrium interest rate in the absence of intervention by the external authority).

Also, contrasting the literature cited above, one of the main contributions of this paper is that we allow explicitly for disequilibrium in the markets caused by the introduction of a fixed interest rate different from the natural rate. To deal with this disequilibrium we shall use the common specification where the quantity traded in each market corresponds to the short side, that is, if, for a given value of the interest rate the market demand is of d units and the market supply is of s units, the quantity traded in that market is d or s units, whether there is an excess supply or demand, respectively.

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[1] See also Andres *et al* (2007) and Laubach *et al* (2007).

We use a model with three representative agents: capitalists, workers and entrepreneurs. Capitalists have the sole function of saving and channel these savings into the credit market where entrepreneurs seek funds in order to finance their purchase of factors of production – namely, labor time. We also assume that labour supply is positively related with the wage level. Let's see what happens, for instance, when the interest rate is fixed on a level below the one of the natural rate. The decline in the interest rate implies, on the part of capitalists, a change in the terms of exchange between present consumption and future consumption in favor of present consumption. Capitalists see therefore reduced the incentive to save, preferring to increase their present consumption. This reduction in savings means a decrease on the amount offered in the credit market. Thus, although the entrepreneurs, due to this reduction in the interest rate, prefer to increase their borrowing, they are constrained by the amount (lower) that appears on the supply side of the credit market and are forced to reduce effectively the price paid for their employment of inputs, namely wages. This reduction in wages decreases the labor supply which, in turn, reduces the output of the economy.

A similar analysis is made for the case where the interest rate is fixed at a level above the one of the natural rate and once again we will see that the level of output is decreased by that kind of policy.

The next section of the paper presents the model analyzing in detail the agents´ decisions. Section 3 determines the general equilibrium of the model without external intervention. Section 4 analyzes the case where the interest rate is set exogenously by an external authority and section 5 concludes.

2. THE MODEL

We consider a model with three representative models: Capitalists, Entrepreneurs and Workers.

The conceptualization of this model is based on Rothbard (2004, ch. 5) and its mathematical formalization is based on Christiano (1991). The distinction between agents is understood to be functional – that is, the same person could be capitalist, entrepreneur and worker at the same time.

At a given moment in time, t, we have the following situation: The capitalist has a given stock of the only good which is produced in the economy, Z_t, from which he can

consume or lend part of it to the representative entrepreneur at a certain interest. We assume that the only good in this economy is perishable, not lasting more than a single period of time. The entrepreneur produces this consumption good hiring working time from the representative worker. We also assume that the production of the consumption good requires a unit period of time and that the entrepreneur has to pay the worker at the beginning of the period (in units of the consumption good) obtaining the produced good only at the end of that same period.

So, for each period of time, we have the following characterization concerning the three agents.

The capitalist, at the beginning of the period, has in his possession a given quantity of the consumption good, Z_t, part of which he consumes, c_t, another part of which he lends to the entrepreneur, $b_t = Z_t - c_t$, at a certain interest rate, r_t. At the end of the period the entrepreneur reimburses the amount lent plus interest to the capitalist. With that borrowed quantity the entrepreneur pays the worker a wage, w_t, for which the worker supplies a certain amount of work, L_t. This working time is used to produce a certain quantity of the consumption good, Y_t, which is obtained at the end of the period. With this quantity the entrepreneur reimburses the quantity borrowed plus interest, $b_t(1+r_t)$ and consumes what is left of it, $d_t = Y_t - b_t(1+r_t)$. As the given stock, Z_t, was totally consumed, the new stock available to the capitalist in the next period, Z_{t+1}, equals $b_t(1+r_t)$. So that, at the beginning of each period of time, the amount Z_t is used in consumption by the capitalist, c_t, the worker, w_t, and the entrepreneur, d_t.

In the next section we formalize each agent's actions.

2.1. CAPITALISTS

The capitalist seeks to maximize his intertemporal utility function which depends only on the consumption realized at each period of time subject to the restriction that the amount consumed plus the amount lent cannot be greater than the stock held at the beginning of the period. As noticed before, the next period's stock equals the amount lent the period before plus interest. We can then describe the intertemporal budget constraint of the capitalist by the following expressions:

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$$c_t + b_t \leq Z_t$$
$$c_{t+1} + b_{t+1} \leq b_t(1 + r_t) = Z_{t+1}.$$

As the capitalist maximizes his utility and this utility depends on the amount of the good consumed, he will exhaust all his amount of Z_t in consumption and/or lending, so that the above expressions can be written as equalities. They can be combined in the following way:

$$Z_{t+1} = (Z_t - c_t)(1 + r_t). \tag{1}$$

The optimization problem of the capitalist is then:

$$\underset{c_t, Z_{t+1}}{Max} \sum_{t=0}^{\infty} \beta^t U(c_t)$$
$$s.t. \tag{P1}$$
$$Z_{t+1} = (Z_t - c_t)(1 + r_t).$$

The parameter β is an intertemporal discount factor and varies between zero and one.

The first order condition (FOC) for the problem (P1) is (see appendix A)[2]:

$$U_c(c_t) = \beta(1 + r_t)U_c(c_{t+1}).$$

In what follows we assume a utility function of the form[3]:

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[2] $U_c(c_t)$ is the derivative oF the utility function with respect to consumption at period t, evaluated at the optimal consumption level for that period.
[3] See, v.g, Obstfeld *et al* (1996. p. 28) for the properties of this utility function.

$$U(c_t) = \frac{c_t^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} , \quad \sigma > 0.$$

The FOC is then written as:

$$\frac{c_{t+1}}{c_t} = \beta^\sigma (1+r_t)^\sigma. \tag{2}$$

2.2. ENTREPRENEURS

The entrepreneur seeks to maximize his profit each period of time:

$$Y_t + b_t - w_t L_t - b_t (1+r_t).$$

As the firm borrows the necessary amount to pay wages, we have the following maximization problem:

$$\underset{Y_t, L_t}{Max} \, Y_t - w_t L_t (1+r_t)$$
$$s.t. \tag{P2}$$
$$Y_t = F(L_t).$$

We assume F to be concave and increasing.

The FOC for this problem is:

$$w_t(1+r_t) = F_L(L_t).$$

Assuming a Constant Returns to Scale production function of the form:

$$Y = F(L_t) = L_t \,,\qquad\qquad (3)$$

the FOC is written as:

$$w_t(1+r_t) = 1 \,. \qquad\qquad (4)$$

2.3. WORKERS

We consider the following utility function:

$$U = c^L - \frac{1}{\gamma}L^{\gamma} \,,\qquad \gamma > 1 \,.$$

The consumption of the worker is denoted by c^L and L is the supply of labour.

The optimization problem of the worker is:

$$\begin{aligned} &\underset{c^L,L}{Max}\; c^L - \frac{1}{\gamma}L^{\gamma}\\ &\quad s.t.\\ &\quad c^L \le wL. \end{aligned}$$

The above restriction binds with equality because we assume the worker consumes all his wages.

The FOC for the above problem is:

$$L = w^{\frac{1}{\gamma-1}} \,.$$

Which gives a labour supply function. It can be shown that $1/(\gamma-1)$ is the elasticity supply of labour. If we consider $\gamma = 2$, that elasticity equals 1. For analytical convenience we use this specification so the supply of labour is:

$$L^S = w. \tag{5}$$

3. GENERAL EQUILIBRIUM

A General Equilibrium for this economy is a sequence of prices: $\{r_t, w_t\}_{t=0}^{\infty}$ and quantities: $\{c_t, c_t^L, Z_{t+1}, Y_t, b_t, L_t, d_t\}_{t=0}^{\infty}$ such that:

1. Given Z_0 and $\{r_t\}_{t=0}^{\infty}$ the representative capitalist solves:

$$\underset{c_t, Z_{t+1}}{Max} \sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} \right), \quad \sigma > 0$$

s.t.

$$Z_{t+1} = b_t^S (1+r_t)$$
$$b_t^S = Z_t - c_t.$$

2. Given $\{r_t, w_t\}_{t=0}^{\infty}$ the representative entrepreneur solves:

$$\underset{Y_t, L_t^D}{Max}\, d_t = Y_t - b_t^D (1+r_t)$$

s.t.

$$Y_t = L_t^D$$
$$b_t^D = w_t L_t^D.$$

3. Given $\{w_t\}_{t=0}^{\infty}$ the representative worker solves:

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$$\underset{c_t^L, L_t^S}{Max} \; c_t^L - \frac{1}{2}\left(L_t^S\right)^2$$

$$s.t.$$

$$c_t^L = w_t L_t^S.$$

4. Supply equals Demand in each market and at each period t:

$$Y_t = c_t + c_t^L + d_t,$$
$$L_t^S = L_t^D = L_t,$$
$$b_t^S = b_t^D = b_t.$$

We now seek a steady state for this economy.

From the FOCs of the Capitalist (see equations (1) and (2)), we have:

$$Z = (Z - c)(1 + r),$$
$$1 = \beta^\sigma (1 + r)^\sigma.$$

From the FOCs of the Entrepreneur (see equations (3) and (4)):

$$Y = L,$$
$$w(1 + r) = 1,$$
$$d = 0.$$

From the FOC of the worker (see equation (5)):

$$L = w.$$

From the clearing of the markets condition it follows that:

$$Y = c + wL,$$
$$Z - c = wL.$$

The solution to the above system of equations is:

$$Y = \beta,$$
$$w = \beta,$$
$$c = \beta(1-\beta),$$
$$Z = \beta,$$
$$r = \frac{1}{\beta} - 1 \equiv r^N,$$
$$L = \beta.$$

In Appendix C we show that this steady-state equilibrium is saddle-path stable.

4. EXOGENOUS DETERMINATION OF THE INTEREST RATE

Now, starting from an equilibrium steady-state, where the interest rate is at a level r^N, that we call the natural rate level, we show the effects of an exogenous imposition of the interest rate at a level $\bar{r}$ different from $r^N = \frac{1}{\beta} - 1$.

We further note that when the interest rate is constant and equal to $\bar{r}$, the problem (P1) may be solved in order to obtain a consumption function (in the language of Dynamic Programming, a "policy function"). Thus, at each moment of time we obtain c_t as a function of β, σ and Z_t (see appendix B for details):

$$c_t = \left[1 - \beta^\sigma (1+\bar{r})^{\sigma-1}\right] Z_t. \tag{6}$$

This function tells us that, at each moment, the capitalist consumes a fixed proportion of his stock of the consumption good, Z_t, and lends the remaining part. (Note that the term between the parentheses is a constant number). Note, further, that present consumption varies inversely with the interest rate. That is, for a given amount of the stock of the consumption good and the other two parameters, the greater the interest rate the less will be present consumption.

Now, if we substitute c_t in (6) into the budget constraint (1), we get:

$$Z_{t+1} = \left[Z_t - \left(1 - \beta^\sigma (1+\bar{r})^{\sigma-1}\right) Z_t \right] (1+\bar{r}).$$

Which is the same as:

$$\frac{Z_{t+1}}{Z_t} = \beta^\sigma (1+\bar{r})^\sigma. \tag{7}$$

Notice that for β and σ given, Z_{t+1} equals Z_t (that is, the lhs of (7) equals one) if and only if $\bar{r} = \dfrac{1}{\beta} - 1$. This means that when $\bar{r} \neq r^N$ the stock of the consumption good will either grow or decay. And this depends on the level of $\bar{r}$. That is what we are going now to analyze.

4.1. INTEREST RATE FIXED BELOW THE LEVEL OF THE NATURAL RATE

We first analyze the situation in which the interest rate is fixed below the level of the natural interest, that is, $\bar{r} < \dfrac{1}{\beta} - 1$. The dynamical system consists of equations (6) and (7).

In this case, from (7), $\dfrac{Z_{t+1}}{Z_t} < 1$ for all t and, hence, the stock of the consumption good, Z_t, tends to zero through time. As, from (6), with a fixed interest rate the consumption

level of the capitalist is a constant proportion of Z_t, it follows that c_t also decays through time tending asymptotically to zero.

On the other hand, the amount the capitalist lends to the entrepreneur is $Z_t - c_t$. But, as we have just seen, as the values of those two variables tend to zero, so the amount lent tends also to zero. In this way the total wages paid by the entrepreneur to the worker (which the first obtains from the amount borrowed to the capitalist) also tends to zero.

As labour supply is positively related with the wage level, that supply also tends to zero and so the output of the economy.

We sum up the present discussion in the following proposition:

PROPOSITION 1: When labour supply is positively related with the wage level, the exogenous imposition of the real interest rate on a level inferior to that of the natural rate has the consequence that the output of the economy diminishes progressively through time.

4.2. INTEREST RATE FIXED ABOVE THE LEVEL OF THE NATURAL RATE

Starting at the steady state, if the interest rate is set at a level above the natural rate, $1/\beta - 1$, which is the equilibrium interest rate at the steady state, the amount supplied by the capitalist in the credit market is greater than the amount demanded by the entrepreneur. To see this, note that if r_t rises, then $B_t^s = Z_t - c_t$ also rises, for a given Z_t. This is because, from (6), c_t diminishes when r_t rises for a given Z_t. On the other hand, the amount demanded in the credit market goes down because, from (4) and (5) and assuming equilibrium in the labour market,

$$B_t^D = w_t L_t = \left(\frac{1}{1+\bar{r}}\right)^2.$$
(8)

This disequilibrium is resolved in the usual way, that is, the transaction that takes place corresponds to the shortest amount in the market. So, the entrepreneur takes in the credit market the amount he really intends to take and pays a wage of:

$$w_t = \frac{1}{1+\bar{r}}. \tag{9}$$

On the other hand, assuming the consumption good is perishable, the capitalist consumes the excess amount that he offers in the credit market. Profit continues to be zero.

From now on, and while $r_t = \bar{r}$, Z_t is constant because

$$Z_{t+1} = B_t^D (1+\bar{r}) = \left(\frac{1}{1+\bar{r}}\right)^2 (1+\bar{r}) = \frac{1}{1+\bar{r}}. \tag{10}$$

We further show that while the interest rate is above the natural rate the supply in the credit market exceeds the demand. This is because from (6) and (10):

$$c_t = \frac{1}{1+\bar{r}} - \beta^\sigma (1+\bar{r})^{\sigma-2}. \tag{11}$$

And then B^S is given, from (10) and (11) by:

$$B^S = Z_t - c_t = \beta^\sigma \left((1+\bar{r})\right)^{\sigma-2} \tag{12}$$

So $B^s > B^D$ if, from (8) and (12):

$$\beta^\sigma \left((1+\bar{r})\right)^{\sigma-2} > \left(\frac{1}{1+\bar{r}}\right)^2.$$

Which is always true if $\bar{r} > \dfrac{1}{\beta} - 1 = r^N$.

So, the capitalist consumes the supply in excess in the credit market and his total consumption is given by:

$$c_t = Y_t - w_t L_t. \tag{13}$$

Now, from (9), with $\bar{r} > r^N$, the wage rate is lower. From (5) and (3), labour supply and output go down. The output level is now:

$$Y = L = w = \frac{1}{1 + \bar{r}} < \frac{1}{1 + r^N}. \tag{14}$$

Using (14) in (13) we obtain the consumption level of the capitalist when $\bar{r} > r^N$:

$$c_t = \frac{1}{1 + \bar{r}} \left(1 - \frac{1}{1 + \bar{r}} \right). \tag{15}$$

In order to see if the consumption of capitalists rises, diminishes or stays the same following the imposition of the interest rate at a level above r^N, we take the derivative of c_t with respect to $\bar{r}$ in (15).

$$\frac{\partial c_t}{\partial \bar{r}} = \frac{1}{\left(1 + \bar{r}\right)^2} \left(-1 + \frac{2}{1 + \bar{r}} \right). \tag{16}$$

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From (16) it follows that $\dfrac{\partial c_t}{\partial \bar{r}} > 0$ for all $r^N \le \bar{r} < 1$.

That is, though the output of the economy diminishes, the consumption of the capitalist rises.

We now sum up the above discussion in the following proposition.

PROPOSITION 2: With labor supply positively related to the wage level, the imposition of an exogenous real interest rate exceeding the value of the natural rate of interest causes a decrease in the total output of the economy (for a positive level well defined), a decrease in real wages and an increase (in absolute terms) of the consumption of the capitalists. If the exogenous rate is set to a level above 100% this will also cause a decreasing of the consumption of the capitalists.

5. CONCLUSION

We have studied the consequences of the fixation of the interest rate at a level different from the general equilibrium interest rate level in a setting of a model without money and with output positively related to the wage level.

Our main findings are that that if the interest rate is set on a level below the natural rate of interest this makes the output level decrease gradually over time; on the other hand, setting the interest rate on a level higher than the natural rate of interest makes the output level decrease until it reaches a stationary value, and increases (in normal circumstances) the income of the capitalists at the expense of workers' earnings.

APPENDICES

A. Solution of the optimization problem of the Capitalist.

The capitalist solves problem (P1).

We form the lagrangean (see Ljunqvist et al. (2000), p. 321):

$$L = \sum_{t=0}^{\infty} \beta^t \left[U(c_t) - \lambda_t \left(Z_{t+1} - (Z_t - c_t)(1+r_t) \right) \right].$$

The capitalist chooses $\{c_t, Z_{t+1}\}_{t=0}^{\infty}$. The lagrangean might be written as:

$$L = ... + \beta^t \left(U(c_t) - \lambda_t \left(Z_{t+1} - (Z_t - c_t)(1+r_t) \right) \right) +$$
$$\beta^{t+1} \left(U(c_{t+1}) - \lambda_{t+1} \left(Z_{t+2} - (Z_{t+1} - c_{t+1})(1+r_{t+1}) \right) \right) + ...$$

First order conditions are:

$$\frac{\partial L}{\partial c_t} = U_c(c_t) - \lambda_t (1+r_t) = 0$$

$$\frac{\partial L}{\partial Z_{t+1}} = -\beta^t \lambda_t + \beta^{t+1} \lambda_{t+1}(1+r_{t+1}) = 0.$$

Which is the same as:

$$U_c(c_t) = \lambda_t (1+r_t)$$
$$\lambda_t = \beta \lambda_{t+1}(1+r_{t+1}).$$

Combining the two equations above we obtain:

$$U_c(c_t) = \beta(1+r_t)U_c(c_{t+1}).$$

B. Consumption Function of the Capitalist when the Interest Rate is constant.

We use here the method of undetermined coefficients. See, for example, Chow (1997, p. 33).[4]

The first order condition of the capitalist for problem (P1), with constant interest rate is:

$$c_{t+1} = c_t \beta^\sigma (1+r)^\sigma.$$

We now conjecture that the level of consumption which solves the equation above equals ΩZ_t, where Ω is a constant to be determined. In that case we can rewrite the equation above as:

$$\Omega Z_{t+1} = \Omega Z_t \beta^\sigma (1+r)^\sigma.$$

Substituting the budget constraint (equation (1)) in Z_{t+1} above we have:

$$\Omega(Z_t - c_t)(1+r) = \Omega Z_t \beta^\sigma (1+r)^\sigma.$$

Substituting ΩZ_t for c_t and dividing both sides of the equation by ΩZ_t, we get:

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[4] Compare with Obstfled *et al* (1996, pp. 718-721).

$$1 + r - \Omega(1+r) = \beta^\sigma (1+r)^\sigma.$$

Hence:

$$\Omega = 1 - \beta^\sigma (1+r)^{\sigma-1}.$$

It follows, from $c_t = \Omega Z_t$, that the consumption function is:

$$c_t = \left[1 - \beta^\sigma (1+r)^{\sigma-1} \right] Z_t.$$

This result can be confirmed by inserting this last expression on both sides of the first equation of this appendix and verifying that they are equal.

C. Transitional Dynamics.

We now analyze how the system evolves to the steady state.

The budget constraint of the capitalist says that:

$$Z_t = c_t + b_t.$$

So that, from the restriction of the Entrepreneur:

$$Z_t = c_t + w_t L_t.$$

Now, using (4) and (5) above we have:

$$Z_t = c_t + \left(\frac{1}{1+r_t}\right)^2.$$

Solving for r_t we get:

$$1+r_t = \left(\frac{1}{Z_t - c_t}\right)^{\frac{1}{2}}. \qquad\qquad (C.1)$$

Using the above expression in (2) we get the dynamic equation for c_t:

$$c_{t+1} = c_t \left(\frac{\beta^2}{Z_t - c_t}\right)^{\frac{\sigma}{2}}. \qquad\qquad (C.2)$$

Next, from (1) and (C.1) we get the dynamic equation for Z_t:

$$Z_{t+1} = (Z_t - c_t)^{\frac{1}{2}}. \qquad\qquad (C.3)$$

Equations (C.2) and (C.3) form a planar dynamic system which can be linearized around the steady-state in the following way (Fuente, (2000), p.487):

$$\begin{bmatrix} c_{t+1} - c \\ Z_{t+1} - Z \end{bmatrix} = J \begin{bmatrix} c_t - c \\ Z_t - Z \end{bmatrix}. \qquad\qquad (C.4)$$

In the system above c and Z are the steady-state values of c_t and Z_t respectively, β and $\beta(1-\beta)$, and J is the jacobian matrix evaluated at the steady-state levels:

$$J = \begin{pmatrix} \dfrac{\partial c_{t+1}}{c_t} & \dfrac{\partial c_{t+1}}{Z_t} \\ \dfrac{\partial Z_{t+1}}{c_t} & \dfrac{\partial Z_{t+1}}{Z_t} \end{pmatrix}.$$

The eigenvalues of the jacobian inform us about the kind of stability of the system. Below we obtain the eigenvalues of the matrix J with the aid of MATHEMATICA (see Wolfram, 2003). It is there shown that for reasonable values for the parameters β and σ one of the eigenvalues is less than one and the other greater than one. In this case we say that the above system is saddle-path stable, meaning that for a given Z_t there is only one c_t for which there is a trajectory that leads to the steady state; in other words, the solution to the system is unique.

In[1]:= g1 = D[o (β^2/ (z - o))^(σ/2), o] (* Derivative of o_{b+1} w.r.t. o_b *)

Out[1]= $\left(\frac{\beta^2}{-\sigma+z}\right)^{\sigma/2} + \frac{\sigma \beta^2 \left(\frac{\beta^2}{-\sigma+z}\right)^{-1-\frac{\sigma}{2}} \sigma}{2 (-\sigma+z)^2}$

In[2]:= g1 /. {z → β, σ → β (1 - β)} // FullSimplify

Out[2]= $\frac{1}{2} \left(2 + \left(-1 + \frac{1}{\beta}\right) \sigma\right)$

In[3]:= g2 = D[o (β^2/ (z - o))^(σ/2), z] (* Derivative of o_{b+1} w.r.t. z_b *)

Out[3]= $-\frac{\sigma \beta^2 \left(\frac{\beta^2}{-\sigma+z}\right)^{-1-\frac{\sigma}{2}} \sigma}{2 (-\sigma+z)^2}$

In[4]:= g2 /. {z → β, σ → β (1 - β)} // FullSimplify

Out[4]= $\frac{(-1 + \beta) \sigma}{2 \beta}$

In[5]:= g3 = D[(z - o)^(1/2), o] (* Derivative of z_{b+1} w.r.t. o_b *)

Out[5]= $-\frac{1}{2\sqrt{-\sigma+z}}$

In[6]:= g3 /. {z → β, σ → β (1 - β)} // FullSimplify

Out[6]= $-\frac{1}{2\sqrt{\beta^2}}$

In[7]:= g4 = D[(z - o)^(1/2), z] (* Derivative of z_{b+1} w.r.t. z_b *)

Out[7]= $\frac{1}{2\sqrt{-\sigma+z}}$

In[8]:= g4 /. {z → β, σ → β (1 - β)} // FullSimplify

Out[8]= $\frac{1}{2\sqrt{\beta^2}}$

In[9]:= J = $\left\{\left\{\frac{1}{2} \left(2 + \left(-1 + \frac{1}{\beta}\right) \sigma\right), \frac{(-1 + \beta) \sigma}{2 \beta}\right\}, \left\{-\frac{1}{2\beta}, \frac{1}{2\beta}\right\}\right\}$; (* J matrix *)

B = Eigenvalues[J] // FullSimplify (* eigenvalues of the system *)

Out[10]= $\left\{\frac{1 + 2\beta + \sigma - \beta\sigma - \sqrt{-8\beta + (1 - \beta(-2+\sigma)+\sigma)^2}}{4\beta}, \frac{1 + 2\beta + \sigma - \beta\sigma + \sqrt{-8\beta + (1 - \beta(-2+\sigma)+\sigma)^2}}{4\beta}\right\}$

In[11]:= B /. {β → 0.95, σ → 2} // FullSimplify (* The values to the eigenvalues when β=0.95 and σ=2 *)

Out[11]= {0.478101, 1.10085}

In[12]:=

β = 0.95; (* Graphs for the possible values to the eigenvalues for different values of σ when β=0.95 *)

Plot$\left[\frac{1 + 2\beta + \sigma - \beta\sigma - \sqrt{-8\beta + (1 - \beta(-2+\sigma)+\sigma)^2}}{4\beta}, \{\sigma, 0, 25\}\right]$ (* graph for the first eigenvalue *)

Plot$\left[\frac{1 + 2\beta + \sigma - \beta\sigma + \sqrt{-8\beta + (1 - \beta(-2+\sigma)+\sigma)^2}}{4\beta}, \{\sigma, 0, 25\}\right]$ (* graph for the first eigenvalue *)

22
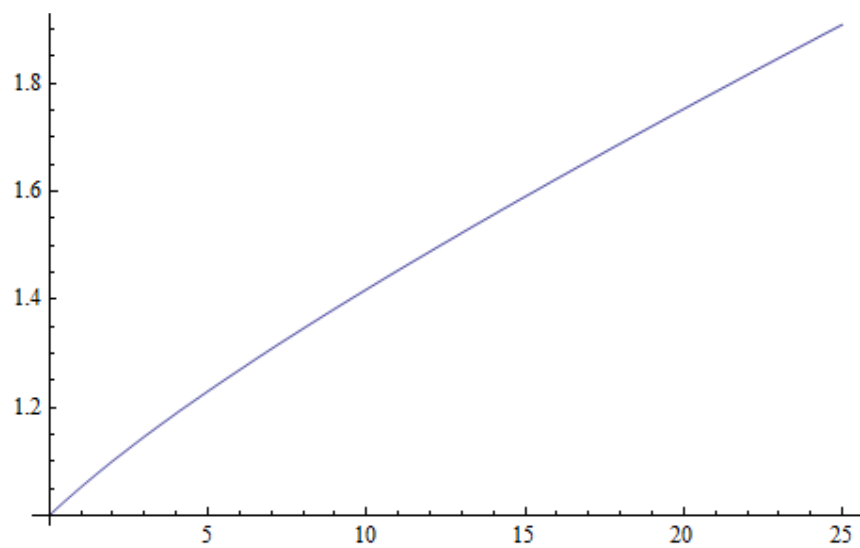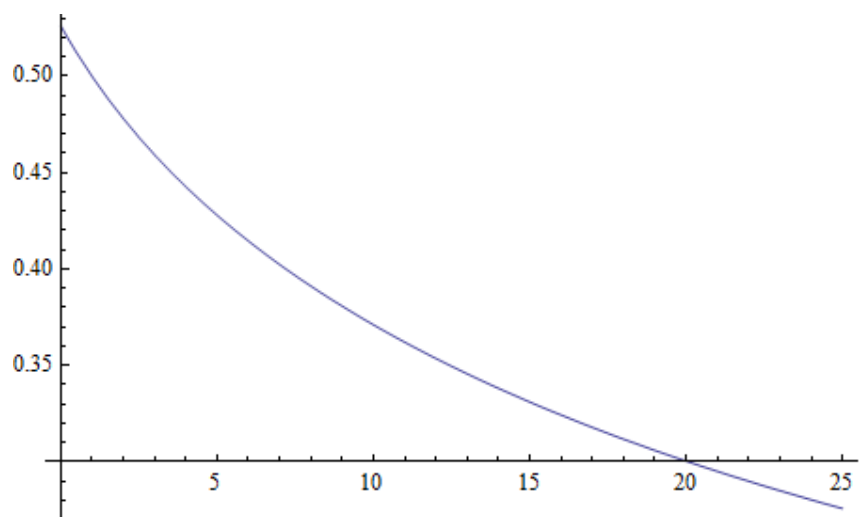
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